

MORIWAKI DIVISORS AND THE AUGMENTED BASE LOCI OF DIVISORS ON THE MODULI SPACE OF CURVES

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ABSTRACT. We study the cone of Moriwaki divisors on \overline{M}_g by means of augmented base loci. Using a result of Moriwaki, we prove that an \mathbb{R} -divisor D satisfies the strict Moriwaki inequalities if and only if $\mathbf{B}_+(D) \subseteq \partial \overline{M}_g$. Then we draw some interesting consequences on the Zariski decomposition of divisors on \overline{M}_g , on the minimal model program of \overline{M}_g and on the log canonical models $\overline{M}_g(\alpha)$.

1. INTRODUCTION

Let $g \geq 3$ and let \overline{M}_g be the moduli space of stable curves on genus g . A striking result of Gibney, Keel and Morrison [GKM, Thm. 0.9] asserts that any nef divisor on \overline{M}_g , not linearly equivalent to zero, must be big. In terms of cones of divisors in the Néron-Severi space $N^1(\overline{M}_g)_{\mathbb{R}}$, this implies that the nef cone does not meet the boundary of the big cone along rational nonzero classes. As a matter of fact, as we shall see, the same is true for real classes: $\text{Nef}(\overline{M}_g) - \{0\} \subset \text{Big}(\overline{M}_g)$. One way to see this is to consider the **Moriwaki cone** $\text{Mor}(\overline{M}_g)$, that is the cone of \mathbb{R} -divisors D on \overline{M}_g that are nef away from the boundary. The cone $\text{Mor}(\overline{M}_g)$ was explicitly described by Moriwaki [M2, Cor. 4.3] in terms of the generators $\lambda, \delta_0, \dots, \delta_{\lfloor g/2 \rfloor}$: an \mathbb{R} -divisor $D \sim a\lambda - b_0\delta_0 - \dots - b_{\lfloor g/2 \rfloor}\delta_{\lfloor g/2 \rfloor}$ belongs to $\text{Mor}(\overline{M}_g)$ if and only if it is an **M-divisor**, that is it satisfies the **Moriwaki inequalities**

$$(1) \quad a \geq 0, \quad a \geq \frac{8g+4}{g}b_0, \quad a \geq \frac{2g+1}{i(g-i)}b_i, \quad \text{for all } i = 1, \dots, \lfloor g/2 \rfloor.$$

The starting idea of this paper is that both the Moriwaki cone and its interior, that is the cone of those \mathbb{R} -divisors that satisfy the strict Moriwaki inequalities and which we call **strict M-divisors**, can be interpreted in terms of restricted and augmented base loci.

Definition 1.1. *Let X be a normal projective variety and let D be an \mathbb{R} -Cartier \mathbb{R} -divisor on X . The **stable base locus** $\mathbf{B}(D)$ of D is X if there is no \mathbb{R} -Cartier \mathbb{R} -divisor $E \geq 0$ such that $E \sim_{\mathbb{R}} D$, or otherwise*

$$\mathbf{B}(D) = \bigcap_{E \geq 0: E \sim_{\mathbb{R}} D} \text{Supp}(E).$$

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The **augmented base locus** and the **restricted base locus** of D are, respectively,

$$\mathbf{B}_+(D) = \bigcap_{A \text{ ample}} \mathbf{B}(D - A) \text{ and } \mathbf{B}_-(D) = \bigcup_{A \text{ ample}} \mathbf{B}(D + A)$$

where A runs among all ample \mathbb{R} -Cartier \mathbb{R} -divisors.

Recall that $\mathbf{B}_-(D) \subseteq \mathbf{B}(D) \subseteq \mathbf{B}_+(D)$ and that D is big if and only if $\mathbf{B}_+(D) \subsetneq X$.

Returning to \overline{M}_g , the main result of this article, where the assertion on $\mathbf{B}_-(D)$ is just a rewriting of [M2, Thm. C], is the following

Theorem 1.

Let $g \geq 3$ and let $D \sim a\lambda - b_0\delta_0 - \dots - b_{\lfloor g/2 \rfloor} \delta_{\lfloor g/2 \rfloor}$ be an \mathbb{R} -divisor on \overline{M}_g . Then

$$\mathbf{B}_-(D) \subseteq \partial \overline{M}_g \text{ if and only if } D \text{ is an } M\text{-divisor,}$$

$$\mathbf{B}_+(D) \subseteq \partial \overline{M}_g \text{ if and only if } D \text{ is a strict } M\text{-divisor.}$$

Now nef non zero divisors are strict M -divisors, as one can easily see intersecting with the F -curves [GKM, Thm. 2.1], therefore the first simple consequence of Theorem 1 is that

$$\text{Nef}(\overline{M}_g) - \{0\} \subset \text{Int}(\text{Mor}(\overline{M}_g)) \subset \text{Big}(\overline{M}_g).$$

Note that this gives another proof on \overline{M}_g , but for \mathbb{R} -divisors, of [GKM, Thm. 0.9].

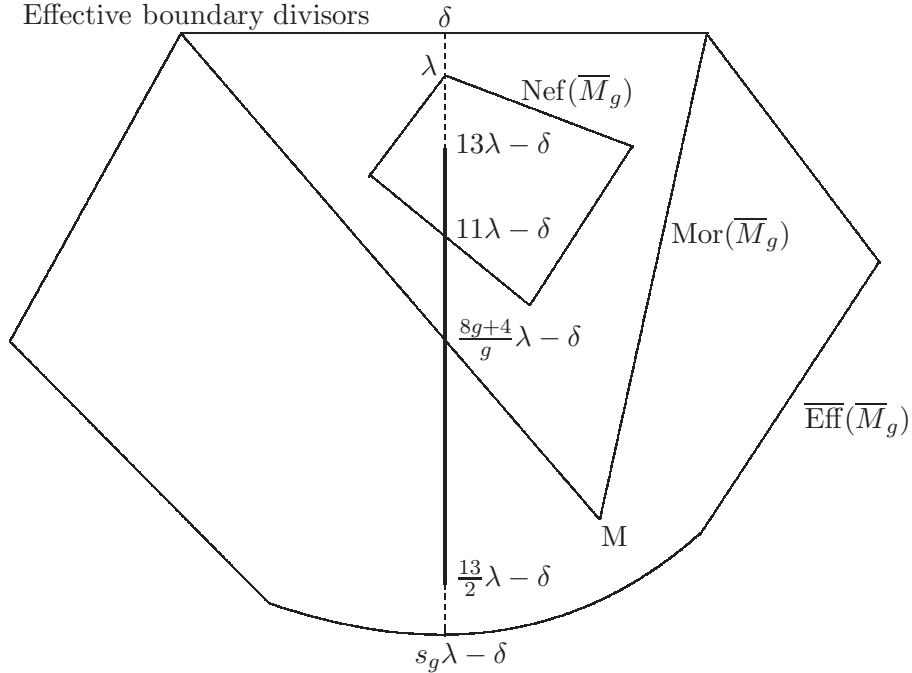


FIGURE 1. A section of the three cones $\text{Nef}(\overline{M}_g) \subseteq \text{Mor}(\overline{M}_g) \subseteq \overline{\text{Eff}}(\overline{M}_g)$ and their intersection with the plane $\langle \lambda, \delta \rangle$. Here s_g is the slope of \overline{M}_g (see [HM]) which, for the sake of the picture, is assumed to be $\leq \frac{13}{2}$ (this is known to be true for $g \geq 22$ by [EH, Thm. 1 and 2], [F1, Thm. 1] and [F2, Thm. 1.4]).

Remark 1.2. It follows from (1) that the $\text{Mor}(\overline{M}_g)$ is a simplicial polyhedral cone whose extremal rays are generated by the boundary divisors $\{\delta_0, \dots, \delta_{\lfloor g/2 \rfloor}\}$ and by the **Moriwaki divisor**

$$(2) \quad M = (8g + 4)\lambda - g\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} 4i(g - i)\delta_i.$$

Note that M is a big divisor (see Lemma 4.2 in the appendix). This implies that the boundary of $\text{Mor}(\overline{M}_g)$ intersects the boundary of the pseudoeffective cone $\overline{\text{Eff}}(\overline{M}_g)$ only in the common codimension-one face formed by effective boundary divisors.

The Moriwaki divisor also appears in the works of Hain-Reed [HR] and of Hain [Hai].

Another consequence of Theorem 1 is that it gives many compactifications of M_g , generalizing [GKM, Cor. 0.11] (this also holds for $M_{g,n}$, see [CL, Cor. 2]).

Corollary 1.

Let $g \geq 3$, let D be a \mathbb{Q} -divisor on \overline{M}_g such that $\kappa(\overline{M}_g, D) \geq 0$ and for $m \in \mathbb{N}$ consider the map $\varphi_{mD} : \overline{M}_g \dashrightarrow \mathbb{P}H^0(\overline{M}_g, mD)$. If D is a strict M -divisor then there exists $m_0 \in \mathbb{N}$ such that φ_{mm_0D} is an isomorphism over M_g for any $m \in \mathbb{N}$. Viceversa if there exists $m_1 \in \mathbb{N}$ such that φ_{mm_1D} is an isomorphism over M_g for any $m \in \mathbb{N}$, then $\mathbf{B}_+(D) \subseteq \partial\overline{M}_g \cup \mathbf{B}(D)$ and D is a strict M -divisor when $\mathbf{B}(D) \subseteq \partial\overline{M}_g$.

It would be interesting to know whether some of the compactifications obtained in Corollary 1 arise from (stable) modular compactifications in the sense of [S, Def. 1.1, 1.2] or if, conversely, all the (stable) modular compactifications of [S] arise from strict M -divisors.

We can also apply Theorem 1 to get some information on the log canonical models introduced by Hassett and Hyeon [HH1], [HH2],

$$f_\alpha : \overline{M}_g \dashrightarrow \overline{M}_g(\alpha) = \text{Proj} \left(\bigoplus_{m \geq 0} H^0(\overline{M}_g, [m(13\lambda - (2 - \alpha)\delta)]) \right)$$

for $\alpha \in [0, 1] \cap \mathbb{Q}$, where f_α is the standard rational map associated to the construction of Proj (see [ST, Lemma 14.1]), or equivalently, the map associated to the linear system $|m(13\lambda - (2 - \alpha)\delta)|$, for $m \gg 0$ (see Cor. 2.3).

In Figure 1, we have depicted the intersection of the segment $\left\{ \frac{13}{2 - \alpha}\lambda - \delta : \alpha \in [0, 1] \right\}$ with the cones $\text{Nef}(\overline{M}_g) \subseteq \text{Mor}(\overline{M}_g) \subseteq \overline{\text{Eff}}(\overline{M}_g)$.

It has been asked by Hassett¹ whether the map f_α is an isomorphism over M_g when $\alpha > \frac{3g+8}{8g+4}$. We give an affirmative answer in the following

Corollary 2.

Let $g \geq 3$. Then

- (i) f_α is an isomorphism over M_g if and only if $\alpha > \frac{3g+8}{8g+4}$;
- (ii) If $\alpha = \frac{3g+8}{8g+4}$ then f_α is defined over M_g and it contracts the hyperelliptic locus $\overline{H}_g \subset \overline{M}_g$;
- (iii) If $\alpha < \frac{3g+8}{8g+4}$ then the hyperelliptic locus \overline{H}_g is contained in $\mathbf{B}_-(13\lambda - (2 - \alpha)\delta)$.

¹in the open problem session of the AIM workshop “The log minimal model program for the moduli space of curves”, Palo Alto (California, USA), 10-14 December 2012. During the same problem session, M. Fedorchuk said that he could answer to the question away from the hyperelliptic locus.

Note that part (iii) implies that f_α is not defined over \overline{H}_g whenever \overline{H}_g is not contained in a divisorial component of $\mathbf{B}(K_\alpha)$ (which of course can occur only for $g \geq 4$).

Our next goal is deduce, from Theorem 1, some interesting consequences on the Zariski decomposition of divisors and on the minimal models of \overline{M}_g . We first recall the following [C, Def. 1.1], [K1, §1], [M1, Def. 1.1.5]

Definition 1.3. *Let X be a normal projective variety and let D be a pseudoeffective \mathbb{R} -Cartier \mathbb{R} -divisor on X . We say that D has an \mathbb{R} -CKM Zariski decomposition if we can write*

$$D = P + N$$

where P, N are \mathbb{R} -Cartier \mathbb{R} -divisors such that P is nef, N is effective and $h^0(\lfloor mD \rfloor) = h^0(\lfloor mP \rfloor)$ for all $m \in \mathbb{N}$, where $\lfloor mD \rfloor$ and $\lfloor mP \rfloor$ are the round downs.

While on a smooth surface a Zariski decomposition always exists, by the celebrated result of Zariski, in general, on higher dimensional varieties, divisors may or may not have an \mathbb{R} -CKM Zariski decomposition, even if we admit to pass to a birational model [N, Thm. IV.2.10], [Le, Thm. 1.1]. On the other hand, on a variety of nonnegative Kodaira dimension, the canonical bundle is expected to admit an \mathbb{R} -CKM Zariski decomposition, after passing to a birational model, as a consequence of the conjectured existence of minimal models.

On \overline{M}_g we obtain

Corollary 3.

Let $g \geq 3$ and let D be an \mathbb{R} -divisor on \overline{M}_g such that $\kappa(D) \geq 1$. If D has an \mathbb{R} -CKM Zariski decomposition, then D is a strict M -divisor. In particular, when $\kappa(\overline{M}_g) \geq 1$ (currently for $g \geq 22$), the canonical divisor $K_{\overline{M}_g}$ does not have an \mathbb{R} -CKM Zariski decomposition.

We stress that, for $g \geq 24$ or $g = 22$, since $K_{\overline{M}_g}$ is big, the minimal model of \overline{M}_g exists by [BCHM, Lemma 10.1 and Thm. 1.2], whence the pull-back of $K_{\overline{M}_g}$ does have an \mathbb{R} -CKM Zariski decomposition on some birational model of \overline{M}_g . On the other hand \overline{M}_g is an interesting example of a normal projective variety whose canonical bundle does not have an \mathbb{R} -CKM Zariski decomposition.

Corollary 4.

Let g be such that $\kappa(\overline{M}_g) \geq 1$ (currently $g \geq 22$). Then there is no $K_{\overline{M}_g}$ -non positive projective birational morphism $f : \overline{M}_g \rightarrow X$ onto a normal \mathbb{Q} -Gorenstein variety X with K_X nef. In particular if $K_{\overline{M}_g}$ is big (currently for $g \geq 24$ or $g = 22$) and $f : \overline{M}_g \dashrightarrow (\overline{M}_g)_{\min}$ is a rational map to a minimal model obtained via contractions of extremal rays and flips, then f cannot be a morphism, that is, it is not possible to reach a minimal model of \overline{M}_g only via contractions of extremal rays: at some step one must flip.

2. GENERALITIES ON PROJ AND ZARISKI DECOMPOSITION

We collect in this section some general facts that will be used in the proofs. They are all most likely well-known, but we include them for the lack of a reference.

Definition 2.1. *Let X be a normal projective variety, let D be a \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . We set*

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, \lfloor mD \rfloor)$$

for the ring of sections and, if mD is Cartier and $H^0(X, mD) \neq \{0\}$,

$$\varphi_m : X \dashrightarrow Y_m \subseteq \mathbb{P}H^0(X, mD)$$

for the map associated to $|mD|$, where Y_m is the closure of its image (endowed with its reduced scheme structure).

Lemma 2.2. *Let X be a normal projective variety and let D be a \mathbb{Q} -Cartier \mathbb{Q} -divisor on X such that $\kappa(X, D) \geq 0$ and $R(X, D)$ is a finitely generated \mathbb{C} -algebra. Then there is $m_0 \in \mathbb{N}$ such that $Y_{km_0} \cong \text{Proj}(R(X, D))$ is normal for all $k \in \mathbb{N}$. Moreover, with this identification, the standard rational map associated to the construction of Proj (see [ST, Lemma 14.1]), $f_D : X \dashrightarrow \text{Proj}(R(X, D))$ coincides with $\varphi_{km_0} : X \dashrightarrow Y_{km_0}$ for all $k \in \mathbb{N}$.*

Proof. By [EGA2, Prop. 2.4.7(i)] we can assume that D is Cartier. By [EGA2, Lemma 2.1.6(v)] there exists $s \in \mathbb{N}$ such that

$$(3) \quad S^h H^0(X, sD) \rightarrow H^0(X, hsD) \text{ is surjective for all } h \in \mathbb{N}.$$

Since $\kappa(X, D) \geq 0$, we get that $H^0(X, sD) \neq \{0\}$ and that $\mathbf{B}(D) = \text{Bs}(|hsD|)$ for all $h \in \mathbb{N}$. Let

$$\begin{array}{ccc} \tilde{X} & & \\ p \downarrow & \searrow q & \\ X & \xrightarrow[\varphi_s]{} & Y_s \end{array}$$

be a resolution of φ_s , whence with \tilde{X} smooth and p birational. We can write

$$(4) \quad p^*(sD) = M + F$$

with $|M|$ base-point free, F base component of $|p^*(sD)|$ and $p(\text{Supp}(F)) = \mathbf{B}(D)$. Since X is normal and p is birational we have, by Zariski's main Theorem [Har, Proof of Cor. III.11.4], that p is an algebraic fiber space [La1, Def. 2.1.11] and therefore, for all $h \in \mathbb{N}$,

$$(5) \quad H^0(X, hsD) \cong H^0(\tilde{X}, p^*(hsD)).$$

It follows by finite generation that, for all $h \in \mathbb{N}$, hF is the base component of $|p^*(hsD)|$, whence

$$(6) \quad H^0(\tilde{X}, p^*(hsD)) \cong H^0(\tilde{X}, hM).$$

But then $Y_{hs} = \text{Im}\{\varphi_{hM} : \tilde{X} \rightarrow \mathbb{P}H^0(\tilde{X}, hM)\}$ for all $h \in \mathbb{N}$. On the other hand, by [La1, Thm. 2.1.27], there is $h_0 \in \mathbb{N}$ and an algebraic fiber space $\phi : \tilde{X} \rightarrow Z$ such that $\varphi_{hM} = \phi$ and $\text{Im } \varphi_{hM} = Z$ for all $h \geq h_0$. Now Z is normal by [La1, Thm. 2.1.15], whence setting $m_0 = h_0 s$ we get that $Y_{km_0} = Z$ is normal for all $k \in \mathbb{N}$.

Let A be an ample divisor on Z such that $h_0 M = \phi^*(A)$. As ϕ is an algebraic fiber space, we get

$$(7) \quad H^0(\tilde{X}, sh_0 M) = H^0(\tilde{X}, \phi^*(sA)) \cong H^0(Z, sA).$$

Since the product is given by multiplication of sections, we deduce by (5), (6) and (7), that $R(X, m_0 D) \cong R(\tilde{X}, p^*(m_0 D)) \cong R(\tilde{X}, h_0 M) \cong R(\tilde{X}, \phi^*(A)) \cong R(Z, A)$. Finally by [EGA2, Prop. 2.4.7(i)] we get $\text{Proj}(R(X, D)) \cong \text{Proj}(R(X, m_0 D)) \cong \text{Proj}(R(Z, A)) \cong Z$ since A is ample.

By [ST, Lemma 14.1], given a graded ring S , a scheme T with a line bundle \mathcal{L} and a homomorphism of graded rings $\psi : S \rightarrow R(T, \mathcal{L})$, there is a morphism

$$f_D : U(\psi) \rightarrow \text{Proj}(R(X, D))$$

where $U(\psi)$ is the union of the open subsets $T_{\psi(f)}$, with $f \in S_d, d > 0$. In our case, setting $T = X, \mathcal{L} = \mathcal{O}_X(D), S = R(X, D)$ and $\psi = \text{Id}_{R(X, D)}$, we have that $U(\psi) = X - \mathbf{B}(D)$ and we get a rational map $f_D : X \dashrightarrow \text{Proj}(R(X, D))$ defined on $X - \mathbf{B}(D)$. On the other hand, for any $d \in \mathbb{N}$ such that $\mathbf{B}(D) = \text{Bs}(|dD|)$, by [ST, Lemma 14.1], we have that f_D coincides on $X - \mathbf{B}(D)$ with the morphism $X - \mathbf{B}(D) \rightarrow \text{Proj}(R(X, D))$ defined on [ST, Lemma 12.1], which, given the immersion $\text{Proj}(R(X, D)) \subset \mathbb{P}^r, r = h^0(X, dD)$, is just the morphism φ_d . \square

We draw a consequence on the spaces $\overline{M}_g(\alpha)$.

Corollary 2.3. *For every $\alpha \in [0, 1] \cap \mathbb{Q}$ we have that $\overline{M}_g(\alpha)$ is normal and the rational map $f_\alpha : \overline{M}_g \dashrightarrow \overline{M}_g(\alpha)$ is given by $\varphi_{m(13\lambda - (2-\alpha)\delta)}$ for $m \gg 0$ and divisible.*

Proof. Set $K_\alpha = 13\lambda - (2 - \alpha)\delta$. We can assume that $\kappa(K_\alpha) \geq 0$. If $\alpha = 1$ the assertion follows by [CH, Thm. 1.3], as $13\lambda - \delta$ is ample. Now assume $\alpha < 1$ and set $B_\alpha = \alpha(\Delta_0 + \Delta_2 + \dots + \Delta_{\lfloor g/2 \rfloor}) + \frac{\alpha+1}{2}\Delta_1$, so that $K_\alpha = K_{\overline{M}_g} + B_\alpha$ and $(\overline{M}_g, B_\alpha)$ is klt by [HH1, Proof of Prop. A.13] or [BCHM, Proof of Cor. 1.2.1]. Then $R(\overline{M}_g, K_\alpha)$ is a finitely generated \mathbb{C} -algebra by [BCHM, Cor. 1.1.2] and we just apply Lemma 2.2. \square

We also need a result about Zariski decompositions.

Lemma 2.4. *Let X be a normal \mathbb{Q} -factorial projective variety, let D be an \mathbb{R} -divisor on X having an \mathbb{R} -CKM Zariski decomposition $D = P + N$.*

Then $\mathbf{B}_+(D) = \mathbf{B}_+(P)$ and $\text{Supp}(N) \subseteq \mathbf{B}_+(D)$.

Proof. If D is not big then P is also not big, so that $\mathbf{B}_+(D) = \mathbf{B}_+(P) = X$.

Suppose now that D is big, so that P is also big by [N, Thm. II.3.7 and Lemma II.3.16]. We will now use some results in [N, III.1]. We point out that, even though in [N, III.1] they are proved for smooth varieties, the results hold with minor modifications on a normal \mathbb{Q} -factorial projective variety. Given any prime divisor Γ on X , one can define, as in [N, Def. III.1.1],

$$\sigma_\Gamma(D) = \inf\{\text{ord}_\Gamma(E), E \text{ effective } \mathbb{R}\text{-divisor on } X \text{ such that } E \equiv D\}$$

and, for any pseudoeffective \mathbb{R} -divisor F on X , as in [N, Def. III.1.6],

$$\sigma_\Gamma(F) = \lim_{\varepsilon \rightarrow 0^+} \sigma_\Gamma(F + \varepsilon A)$$

where A is an ample divisor (the definition does not depend on the choice of A). Now set

$$N_\sigma(D) = \sum_{\Gamma} \sigma_\Gamma(D) \Gamma, \quad P_\sigma(D) = D - N_\sigma(D).$$

Note that $N_\sigma(D)$ is an \mathbb{R} -divisor by [N, Cor. III.1.11]. The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the σ -decomposition of D (see [N, Def. III.1.12]). By [K2, Lemma 2.6] (or by [N, Rmk. III.1.17(3)]), it follows that $P = P_\sigma(D)$ and $N = N_\sigma(D)$. We also recall that $\text{Supp}(N_\sigma(D)) \subseteq \mathbf{B}(D)$ (see [BBP, Lemma 1.6]).

Given any ample \mathbb{R} -divisor A on X such that $A \leq D$, we find, by [N, Lemma III.1.4], that $\sigma_\Gamma(D) \leq \sigma_\Gamma(D - A) + \sigma_\Gamma(A) = \sigma_\Gamma(D - A) \leq \text{ord}_\Gamma(D - A)$, whence $D - A \geq N$. Therefore

$$\begin{aligned} \mathbf{B}_+(D) &= \bigcap_{A \leq D} \text{Supp}(D - A) = \bigcap_{A \leq D} (\text{Supp}(D - A - N) \cup \text{Supp}(N)) = \\ &= \text{Supp}(N) \cup \bigcap_{A \leq P} \text{Supp}(P - A) = \text{Supp}(N) \cup \mathbf{B}_+(P). \end{aligned}$$

Now let Γ be a prime divisor in the support of N , so that $\sigma_\Gamma(D) > 0$. We will prove that $\Gamma \subseteq \mathbf{B}_+(P)$. Let H be an ample Cartier divisor such that $H - \Gamma$ is ample. Then there exists $\varepsilon > 0$ sufficiently small such that $\varepsilon \leq \sigma_\Gamma(D)$, $\mathbf{B}_+(P) = \mathbf{B}(P - \varepsilon(H - \Gamma))$ by [ELMNP, Prop. 1.5] (note that it is not needed that $P - \varepsilon(H - \Gamma)$ is a \mathbb{Q} -divisor) and $P - \varepsilon H$ is big. By [N, Lemmas III.1.8 and III.1.4] we get

$$0 < \varepsilon = \sigma_\Gamma(P + \varepsilon\Gamma) \leq \sigma_\Gamma(P - \varepsilon(H - \Gamma)) + \sigma_\Gamma(\varepsilon H) = \sigma_\Gamma(P - \varepsilon(H - \Gamma))$$

so that $\Gamma \subseteq \mathbf{B}(P - \varepsilon(H - \Gamma)) = \mathbf{B}_+(P)$. \square

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. We begin by recalling some results of Moriwaki [M2]. In [M2, Lemma 4.1], Moriwaki showed that there exist curves $C, C_0, \dots, C_{\lfloor g/2 \rfloor}$ inside \overline{M}_g , not entirely contained in the boundary $\partial\overline{M}_g$, with the following properties:

- C is contained inside M_g ;
- C_0 is contained in \overline{H}_g and intersects $\partial\overline{M}_g$ in points corresponding to isomorphism classes of irreducible curves with a single node;
- For every $i = 1, \dots, \lfloor g/2 \rfloor$, C_i is contained in \overline{H}_g and intersects $\partial\overline{M}_g$ in points corresponding to isomorphism classes of stable curves formed by two irreducible components of genus i and $g - i$ meeting in a single node;

It follows from the proof of [M2, Prop. 4.2] that the cone spanned by $C, C_0, \dots, C_{\lfloor g/2 \rfloor}$ inside $N_1(\overline{M}_g)_\mathbb{R}$ is the dual of the cone of M-divisors.

Consider now an \mathbb{R} -divisor D on \overline{M}_g such that $\mathbf{B}_-(D) \subseteq \partial\overline{M}_g$ (respectively $\mathbf{B}_+(D) \subseteq \partial\overline{M}_g$) and let γ be one of the curves $C, C_0, \dots, C_{\lfloor g/2 \rfloor}$. Since $\gamma \not\subseteq \partial\overline{M}_g$, we get that $\gamma \not\subseteq \mathbf{B}_-(D)$ (respectively $\gamma \not\subseteq \mathbf{B}_+(D)$) and therefore $D \cdot \gamma \geq 0$ (respectively $D|_\gamma$ is big, that is $D \cdot \gamma > 0$). This shows that D is an M-divisor (respectively a strict M-divisor).

Vice versa suppose first that D is an M-divisor. Now it is easily seen that there exists $\beta \geq 0$ and an effective \mathbb{R} -divisor E on \overline{M}_g such that $D = \beta M + E$ and $\text{Supp}(E) \subseteq \partial\overline{M}_g$, where M is the Moriwaki divisor as in (2).

We recall that the content of [M2, Thm. B] is exactly that $\mathbf{B}_-(M) \subseteq \partial\overline{M}_g$; hence

$$\mathbf{B}_-(D) \subseteq \mathbf{B}_-(M) \cup \text{Supp}(E) \subseteq \partial\overline{M}_g.$$

Moreover, if D is a strict M-divisor, we can choose a sufficiently small ample \mathbb{R} -divisor A on \overline{M}_g such that $D' := D - 2A$ is still a strict M-divisor and $\mathbf{B}_+(D) = \mathbf{B}(D - A)$ by [ELMNP, Prop. 1.5] (note that it is not needed that $D - A$ is a \mathbb{Q} -divisor). Then there exists $\beta' > 0$ and an effective \mathbb{R} -divisor E' on \overline{M}_g such that $D' = \beta' M + E'$ and $\text{Supp}(E') \subseteq \partial\overline{M}_g$. Hence

$$\mathbf{B}_-(D') \subseteq \mathbf{B}_-(M) \cup \text{Supp}(E') \subseteq \partial\overline{M}_g$$

therefore also

$$\mathbf{B}_+(D) = \mathbf{B}(D - A) = \mathbf{B}(D' + A) \subseteq \mathbf{B}_-(D') \subseteq \partial\overline{M}_g.$$

\square

We note that, for some divisors, we can compute exactly the augmented base locus.

Proposition 3.1. *Let $g \geq 3$ and let $D \sim a\lambda - b_0\delta_0 - \dots - b_{\lfloor g/2 \rfloor}\delta_{\lfloor g/2 \rfloor}$ be a big \mathbb{R} -divisor on \overline{M}_g with $b_i \leq 0$ for all $i = 0, \dots, \lfloor g/2 \rfloor$. Then $\mathbf{B}_+(D) = \partial\overline{M}_g$. Moreover if D is a \mathbb{Q} -divisor then, for $m \gg 0$ sufficiently divisible, φ_{mD} is the Torelli morphism to the Satake normal compactification $\overline{M}_g^S := \text{Proj}(R(\overline{M}_g, \lambda))$ of M_g .*

Proof. Recall that λ is semiample, whence, by [La1, Thm. 2.1.15 and 2.1.27], we get an algebraic fiber space $\pi = \varphi_{m\lambda} : \overline{M}_g \rightarrow \text{Im } \phi_{m\lambda} \cong \overline{M}_g^S$ for $m \gg 0$ sufficiently divisible (this is the Torelli morphism to the Satake compactification) and that \overline{M}_g^S is normal. Moreover, it is well-known that $\text{Exc}(\pi) = \partial \overline{M}_g$ (see e.g. [ACG, Chap. XIV, §5]).

In particular, as D is big, this shows that $a > 0$. Let A be an ample \mathbb{Q} -divisor such that $\lambda = \pi^* A$ and set $F = -b_0 \delta_0 - \dots - b_{\lfloor g/2 \rfloor} \delta_{\lfloor g/2 \rfloor}$, so that F is effective and π -exceptional. As \overline{M}_g and \overline{M}_g^S are normal and π is birational, we can apply [BBP, Prop. 2.3]:

$$\mathbf{B}_+(D) = \mathbf{B}_+(\pi^*(aA) + F) = \pi^{-1}(\mathbf{B}_+(aA)) \cup \text{Exc}(\pi) = \text{Exc}(\pi) = \partial \overline{M}_g.$$

Now if D is a \mathbb{Q} -divisor and $m \gg 0$ is such that mD and $ma\lambda$ are Cartier, then

$$H^0(\overline{M}_g, ma\lambda) \cong H^0(\overline{M}_g, m(a\lambda + F)) \cong H^0(\overline{M}_g, mD)$$

and the last assertion of the Proposition follows. \square

Proof of Corollary 1. By [BCL, Thm. 2], given a big \mathbb{Q} -divisor D on \overline{M}_g , we have that there exists $m_0 \in \mathbb{N}$ such that $\overline{M}_g - \mathbf{B}_+(D)$ is the largest open subset of $\overline{M}_g - \mathbf{B}(D)$ where the maps $\varphi_{mm_0 D}$ are an isomorphism for every $m \in \mathbb{N}$. Using this, Corollary 1 follows from Theorem 1. \square

Proof of Corollary 2. Note that $K_\alpha := 13\lambda - (2 - \alpha)\delta$ is a strict M-divisor if and only if $\alpha > \frac{3g+8}{8g+4}$. Then (i) follows from Corollaries 1 and 2.3.

Assume now that $\alpha = \frac{3g+8}{8g+4}$. Then, K_α is a (non-strict) M-divisor and, moreover, it is big, for his slope $s(K_\alpha) = 8 + \frac{4}{g}$ is larger than the one of a Brill-Noether divisor if $g+1$ is not prime (see [EH, Thm. 1]) or of the Petri divisor if g is even (see [EH, Thm. 2]). Also we claim that $\mathbf{B}(K_\alpha) = \mathbf{B}_-(K_\alpha)$. Let $x \in \mathbf{B}(K_\alpha)$ and let v be any divisorial valuation with center $\{x\}$. By the finite generation of $R(\overline{M}_g, K_\alpha)$, as in [ELMNP, Prop. 2.8] or [BBP, §2.2], we have that $v(\|K_\alpha\|) > 0$, whence $x \in \mathbf{B}_-(K_\alpha)$ and the claim is proved. By Theorem 1, we get $\mathbf{B}(K_\alpha) = \mathbf{B}_-(K_\alpha) \subseteq \partial \overline{M}_g$, whence that f_α is defined over M_g .

In order to prove the second statement of (ii), observe that K_α is proportional to the Cornalba-Harris divisor $(8g+4)\lambda - g\delta$ of [CH, Prop. 4.3]. It follows from [CH, Prop. 4.3, Thm. 4.12] that K_α intersects to zero the curves constructed by Cornalba-Harris in [CH, p. 469]²: these are curves in \overline{H}_g given by a family $\pi : X \rightarrow T$ of stable hyperelliptic curves over a smooth projective curve T obtained as a double cover $\eta : X \rightarrow T \times \mathbb{P}^1$ branched over a general curve $C \subset T \times \mathbb{P}^1$ of class $(2g+2, 2m)$ for some $m \geq 1$. As the image of $T \rightarrow \overline{H}_g$ passes through the general point of \overline{H}_g , it follows that the map f_α contracts the hyperelliptic locus $\overline{H}_g \subset \overline{M}_g$. This finishes the proof of (ii).

Assume finally that $\alpha < \frac{3g+8}{8g+4}$. Then K_α intersects negatively the Cornalba-Harris curves considered above, which therefore must belong to $\mathbf{B}_-(K_\alpha)$. By what we said above, we deduce that $\overline{H}_g \subset \mathbf{B}_-(K_\alpha)$, which proves (iii). \square

Proof of Corollary 3. Suppose that $D = P + N$ is an \mathbb{R} -CKM Zariski decomposition. Then P is nef and non trivial, because $\kappa(P) = \kappa(D) \geq 1$, whence it is a strict M-divisor (just intersect with the F -curves [GKM, Thm. 2.1]). Therefore, by Lemma 2.4 and Theorem 1, we have $\mathbf{B}_+(D) = \mathbf{B}_+(P) \subseteq \partial \overline{M}_g$, so that D is a strict M-divisor again by Theorem 1. To

²Indeed, it is easily checked, by [CH, Prop. 4.7], that these curves are all numerically proportional to the curve C_0 constructed in [M2, Lemma 4.1].

conclude we just note that

$$K_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \dots - 2\delta_{[g/2]}$$

is not an M-divisor. \square

Proof of Corollary 4. Let $a \in \mathbb{N}$ be such that $aK_{\overline{M}_g}$ and aK_X are Cartier. Now non-positivity of f means that we have

$$(8) \quad aK_{\overline{M}_g} = f^*(aK_X) + E$$

with $E \geq 0$ and f -exceptional. Setting $P = f^*(aK_X)$ and $N = E$, we see immediately that (8) is an \mathbb{R} -CKM Zariski decomposition of $aK_{\overline{M}_g}$, thus contradicting Corollary 3. To conclude the proof recall that if $K_{\overline{M}_g}$ is big, as we said in the introduction, \overline{M}_g has a minimal model $(\overline{M}_g)_{min}$. Hence $(\overline{M}_g)_{min}$ has normal \mathbb{Q} -factorial dlt singularities, $K_{(\overline{M}_g)_{min}}$ is nef and there is a projective birational map $f : \overline{M}_g \dashrightarrow (\overline{M}_g)_{min}$ that is $K_{\overline{M}_g}$ -negative (in fact f is obtained via contractions of extremal rays and flips). Then f cannot be a morphism, by what we proved above. \square

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4. APPENDIX: THE BIGNESS OF MORIWAKI'S DIVISOR

Given the cumbersome calculations we give, in this appendix, the proof of the bigness of the Moriawaki divisor. We remark that this is just for completeness' sake, as we do not need this fact in the article.

We will use the following

Criterion 4.1. *Let $g \geq 3$ and let $D \equiv \alpha\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$ be an \mathbb{R} -divisor on \overline{M}_g with $a > 0$.*

Assume that there exists an effective \mathbb{R} -divisor $E \equiv \alpha\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} \beta_i \delta_i$ such that

$$(A) \quad \alpha > 0$$

$$(B_i) \quad \beta_i > 0, \text{ for all } 0 \leq i \leq \lfloor g/2 \rfloor$$

and

$$(C_i) \quad \alpha b_i < a \beta_i, \text{ for all } 0 \leq i \leq \lfloor g/2 \rfloor.$$

Then D is big.

Proof. We can choose $v \in \mathbb{R}$, $v \geq 0$ such that, for all $0 \leq i \leq \lfloor g/2 \rfloor$, we have

$$\frac{b_i}{\beta_i} \leq v < \frac{a}{\alpha}.$$

Now $D \equiv (a - v\alpha)\lambda + vE + \sum_{i=0}^{\lfloor g/2 \rfloor} (v\beta_i - b_i)\delta_i$ is big since λ is big. \square

Lemma 4.2. *Let $g \geq 3$ and let $M = (8g + 4)\lambda - g\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} 4i(g - i)\delta_i$ be the Moriwaki divisor on \overline{M}_g . Then M is big.*

Proof. We apply Criterion 4.1. As $a = 8g + 4 > 0$, $b_0 = g > 0$ and, for all $1 \leq i \leq \lfloor g/2 \rfloor$, $b_i = 4i(g - i) > 0$, we will need to verify only (A) and all (C_i) 's.

If $g + 1$ is not prime, as in [EH, Theorem 1], we can write $g + 1 = (r + 1)(s - 1)$, for some integers $s \geq 3$ and $r \geq 1$ and we can consider the Brill-Noether divisor D_s^r on \overline{M}_g . By [EH, Theorem 1] there exists $c > 0$ such that

$$0 \leq \frac{1}{c}D_s^r \equiv (g + 3)\lambda - \frac{g + 1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g - i)\delta_i.$$

Setting $E = \frac{1}{c}D_s^r$, we have that (A) is satisfied. Also (C_0) is equivalent to $g^2 - 3g + 2 > 0$, while, for $i \geq 1$, (C_i) is equivalent to $g - 2 > 0$, so all the (C_i) 's are also satisfied.

Assume from now on that $g + 1$ is prime, so that we can write $g = 2(d - 1)$, for some $d \geq 3$ and we can consider the Petri divisor E_d^1 on \overline{M}_g . By [EH, Theorem 2] there exists $c > 0$ such that

$$0 \leq \frac{1}{c}E_d^1 = (6d^2 + d - 6)\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} f_i\delta_i$$

where

$$(9) \quad f_0 = d(d - 1);$$

$$(10) \quad f_1 = (2d - 3)(3d - 2);$$

$$(11) \quad f_2 = 3(d - 2)(4d - 3).$$

Moreover, setting $k = d - 1$ and

$$(12) \quad \gamma_i = (i - 1)(i - 2) \frac{(2k - 2)!}{k!(k - 1)!} - \sum_{l=1}^{\lfloor \frac{i-2}{2} \rfloor} 2(i - 1 - 2l) \frac{(2l)!(2k - 2 - 2l)!}{(l + 1)!l!(k - l)!(k - l + 1)!}$$

by [EH, (5.3)] we have

$$(13) \quad f_i = -i(i - 2)f_1 + \frac{i(i - 1)}{2}f_2 + \frac{\gamma_i}{c} \quad \text{for all } 3 \leq i \leq d - 1.$$

Setting $E = \frac{1}{c}E_d^1$ and recalling that $d \geq 3$, we have that (A) is satisfied.

Condition (C_0) is $(6d^2 + d - 6)g < (8g + 4)d(d - 1)$, which is equivalent to $2d^2 - 7d + 6 > 0$, whence it is satisfied.

Condition (C_1) is $(6d^2 + d - 6)4(g - 1) < (8g + 4)(2d - 3)(3d - 2)$, which is equivalent to $2d^3 - 9d^2 + 13d - 6 > 0$, whence it is satisfied.

Condition (C_2) is $(6d^2 + d - 6)8(g - 2) < (8g + 4)3(d - 2)(4d - 3)$, which is equivalent to $24d^3 - 124d^2 + 203d - 102 > 0$, whence it is satisfied.

For all $i = 3, \dots, d-1$, condition (C_i) is equivalent to

$$f_i > \frac{(6d^2 + d - 6)4i(g - i)}{8g + 4} = \frac{(6d^2 + d - 6)i(2d - 2 - i)}{4d - 3}$$

and using (13) can be transformed in

$$(14) \quad -i(i-2)f_1 + \frac{i(i-1)}{2}f_2 + \frac{\gamma_i}{c} > \frac{(6d^2 + d - 6)i(2d - 2 - i)}{4d - 3}, \quad 3 \leq i \leq d-1.$$

To prove (14) we will show that $\gamma_i \geq 0$ for all $i = 3, \dots, d-1$ and

$$(15) \quad -i(i-2)f_1 + \frac{i(i-1)}{2}f_2 > \frac{(6d^2 + d - 6)i(2d - 2 - i)}{4d - 3}, \quad 3 \leq i \leq k.$$

Now (15) is equivalent to

$$i((4d-3)(f_2 - 2f_1) + 2(6d^2 + d - 6)) > (4d-3)(f_2 - 4f_1) + 2(6d^2 + d - 6)(2d-2)$$

and using (10) and (11), to

$$24d^3 - 92d^2 + 109d - 42 > (16d^2 - 47d + 30)i$$

so that, as $i \leq d-1$, we reduce it to $8d^3 - 29d^2 + 32d - 12 > 0$, whence it is satisfied.

It remains to prove that $\gamma_i \geq 0$ for all $i = 3, \dots, k = d-1$.

Note that, by (12), $\gamma_3 = \frac{2(2k-2)!}{k!(k-1)!} > 0$. Hence if we put $c_i = \frac{1}{2}(\gamma_i - \gamma_{i-1})$, we will be done if we show that $c_i \geq 0$ for all $i = 4, \dots, k$. In particular we can suppose $k \geq 4$.

To simplify the notation let

$$b_l = \frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k-l+1)!}$$

so that, by (12), we can write

$$c_i = (i-2) \frac{(2k-2)!}{k!(k-1)!} - \sum_{l=1}^{\lfloor \frac{i-2}{2} \rfloor} b_l.$$

As $c_4 = \frac{(2k-4)!}{k!(k-1)!}(2(2k-2)(2k-3)-1) \geq 0$, setting $d_i = c_i - c_{i-1}$, we are reduced to prove that $d_i \geq 0$ for all $i = 5, \dots, k$.

If i is odd, then $d_i = \frac{(2k-2)!}{k!(k-1)!} \geq 0$, so that we can assume that i is even. In particular we can put $i = 2h + 2$, where $2 \leq h \leq \lfloor \frac{k-2}{2} \rfloor$, and we get

$$d_i = \frac{(2k-2)!}{k!(k-1)!} - \frac{(2h)!(2k-2-2h)!}{(h+1)!h!(k-h)!(k-h+1)!}.$$

In this way, after putting $v_h = \frac{(2h)!(2k-2-2h)!}{(h+1)!h!(k-h)!(k-h+1)!}$, we need to prove that

$$(16) \quad \frac{(2k-2)!}{k!(k-1)!} \geq v_h$$

for all $h \in \{2, \dots, \lfloor \frac{k-2}{2} \rfloor\}$ and for all $k \geq 6$.

We now claim that, for $2 \leq h \leq \lfloor \frac{k-2}{2} \rfloor$, we have $v_h \leq \max\{v_2, v_{\lfloor \frac{k-2}{2} \rfloor}\}$.

In fact, for all $h \geq 3$, we can write $v_h - v_{h-1} = C_{k,h}N_{k,h}$, where

$$C_{k,h} = \frac{2(2h-2)!(2k-2h-2)!}{h!(h-1)!(k-h+1)!(k-h)!(h+1)(k-h+2)(k-h+1)} \geq 0$$

for all $3 \leq h \leq \lfloor \frac{k-2}{2} \rfloor, k \geq 6$, and

$$\begin{aligned} N_{k,h} &= (2h-1)(k-h+2)(k-h+1) - (k-h)(2k-2h-1)(h+1) \\ &= -3k^2 + 13kh - 2k - 10h^2 + 6h - 2. \end{aligned}$$

In particular $v_h \leq v_{h-1}$ if and only if $N_{k,h} \leq 0$, if and only if $h \leq k_1 := \frac{13k+6-\sqrt{49k^2+76k-44}}{20}$ or $h \geq k_2 := \frac{13k+6+\sqrt{49k^2+76k-44}}{20}$. Thus the claim follows by noticing that, for all $k \geq 6$, we have $k_2 > \lfloor \frac{k-2}{2} \rfloor$.

Thanks to the claim it suffices to prove that (16) holds for $h = 2$ and $h = \lfloor \frac{k-2}{2} \rfloor$. Since $v_2 = \frac{2(2k-6)!}{(k-2)!(k-1)!}$, we have that (16) holds for $h = 2$.

Suppose $h = \lfloor \frac{k-2}{2} \rfloor$. If k is even, then $k = 2h + 2$, so that (16) is equivalent to

$$\frac{(4h+2)!}{(2h+2)!(2h+1)!} \geq \frac{(2h)!(2h+2)!}{(h+1)!h!(h+2)!(h+3)!}$$

which in turn is verified if and only if

$$a_h := \frac{(4h+2)!h!(h+1)!(h+2)!(h+3)!}{((2h+2)!)^2(2h+1)!(2h)!} \geq 1$$

for all $h \geq 2$. But $a_2 = \frac{10!3!2!}{(6!)^2} \geq 1$, and, for all $h \geq 3$, we have

$$a_h - a_{h-1} = S_h(T_h - 1)$$

where

$$S_h = \frac{(4h-2)!(h-1)!h!(h+1)!(h+2)!}{((2h)!)^2(2h-1)!(2h-2)!} \geq 0$$

for all $h \geq 2$, and

$$T_h = \frac{(4h+1)(4h-1)(h+2)(h+3)}{(2h+1)^2(2h+1)(2h+2)}.$$

An easy computation gives that $T_h \geq 1$ if and only if $56h^3 + 91h^2 + h - 4 \geq 0$, which, in particular, is true for all $h \geq 2$. Thus, for all $h \geq 2$, $a_h \geq a_2 \geq 1$.

If k is odd, then $k = 2h + 3$, and (16) is verified if and only if

$$a'_h := \frac{(4h+4)!h!(h+1)!(h+3)!(h+4)!}{(2h+3)!(2h+2)!(2h)!(2h+4)!} \geq 1$$

for all $h \geq 2$. Again $a'_2 = \frac{11 \cdot 10 \cdot 9}{7} \geq 1$, and

$$a'_h - a'_{h-1} = S'_h(T'_h - 1)$$

where

$$S'_h = \frac{(4h)!(h-1)!h!(h+2)!(h+3)!}{(2h+1)!(2h)!(2h-2)!(2h+2)!} \geq 0$$

for all $h \geq 2$, and

$$T'_h = \frac{(4h+3)(4h+1)(h+3)(h+4)}{2(2h+3)^2(2h-1)(h+2)}$$

so that $T'_h \geq 1$ if and only if $56h^3 + 215h^2 + 207h + 72 \geq 0$, which, in particular, is true for all $h \geq 2$, and we conclude as before. \square

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